Bottom velocity computation method based on depth integrated model without shallow water assumption

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Abstract

In order to compute local scouring around structure together with the computation of large-scale flow phenomena, we have developed the Bottom Velocity Computation (BVC) method, in which depth-averaged horizontal vorticity equation and water surface velocity equation are solved with shallow water equations to evaluate bottom velocity field and vertical velocity distribution. The previous BVC method by the authors has been based on the shallow water assumption, but the validity of the shallow water assumption for local flow structure has not been clarified. This paper presents applicable ranges of 2D, 3D and quasi-3D numerical model by the ratio of representative water depth to representative horizontal scale. Then, the general BVC method based on a depth-integrated model without the shallow water assumption is developed. The method is applied to flow field around a cylinder on flat rough bed measured by Roulund *et al.*(2005).

1. Introduction

Local scouring around structure is one of the most important issues in the hydraulic engineering field. There are many researches on the local scouring. Flow fields around structures inducing local scouring are so complex that many empirical formulae are proposed based on experimental results (e.g., Hoffmans & Verheij, 1997). Recently, we can simulate complex turbulence flow and local scouring by 3D turbulence model (e.g., Roulund *et al.*, 2005). On the other hand, local scours around river structures during flood are affected by the larger scale phenomena of flood flows and bed variations, such as discharge hydrograph, planner shapes of rivers and sand bars. So, to provide countermeasure against local scour around river structures, it is to solve simultaneously not only local 3D flow around structures but also large scale phenomena of flood flows. However, applications of the 3D model are still limited to small scale phenomena such as bed variation in experimental channels, because of large computational time, many memories, and a lot of computational task.

Advanced depth integrated model, improved 2D model, is one of the effective method to compute local scouring around structures with large-scale flow phenomena. Advanced depth integrated models to compute bed variation include secondary flow model (e.g. Nishimoto et al., 1992) and a quasi-3D model (e.g. Fukuoka et al., 1992), in which vertical velocity distributions are computed. Recently, we have developed a new quasi-3D method, the Bottom Velocity Computation (BVC) method, in which depth-averaged horizontal vorticity and water surface velocity equations are computed with shallow water equations to evaluate bottom velocity field and vertical velocity distribution (Uchida & Fukuoka, 2009, 2010, 2011). The previous BVC method has been based on the shallow water assumption, in which representative water depth h_0 is much smaller than representative horizontal scale of the objective phenomena L_0 (shallowness parameter: $\varepsilon_s=h_0/L_0<<1$). However, the validity of the assumption has not been clarified for small scale phenomena, such as local flows around structures.

This paper has a twofold purpose. One is to clarify the suitable computational method for a target phenomenon. The other is to develop the general BVC method based on a depth-integrated model without the shallow water assumption and to verify the applicability of the method to flow field around a cylinder measured by Roulund *et al.*(2005).

2. Previous Bottom Velocity Computational Method with Shallow Water Assumption

BVC method has been developed based on equation (1), which is derived by depth integrating vorticity.

$$u_{bi} = u_{si} - \varepsilon_{ij3}\Omega_j h - \frac{\partial Wh}{\partial x_i} + w_s \frac{\partial z_s}{\partial x_i} - w_b \frac{\partial z_b}{\partial x_i} \tag{1}$$

Here, u_{bi}: bottom velocity, u_{si}: water surface velocity, Ω_j: depth-averaged vorticity, h:water depth, w_s, w_b: vertical velocity on water surface and bottom, respectively, z_s: water surface level, z_b: bottom level. For the previous method (Uchida & Fukuoka, 2009, 2010, 2011), bottom velocity is evaluated by water surface velocity and depth-averaged vorticity, neglecting the last three terms in equation (1) by the shallow water assumption, in which the ratio of representative water depth h_0 to representative horizontal scale L_0 is very small $\varepsilon_s=h_0/L_0<<1$ (ε_s: shallowness parameter). Water surface velocity and depth-averaged vorticity are calculated by equation (7) and (10), respectively, together with shallow water equations of continuity equation, momentum equation (2) and the turbulence energy transport equation. And hydrostatic pressure distribution, which is a kind of the shallow water assumption, is assumed to compute depth averaged velocity and water surface velocity equations. Governing equations of the BVC method are set by adding cubic vertical velocity equation (2).

$$u_i = \Delta u_i \left(12\eta^3 - 12\eta^2 + 1\right) + \delta u_i \left(-4\eta^3 + 3\eta^2\right) + U_i \tag{2}$$

Here, U_i: depth averaged velocity, δu_i: $u_{si} - u_{bi}$, Δu_i: $u_{si} - U_i$.

3. Applicability Ranges and Limitations of Calculation Models by Shallowness

3.1 Depth averaged velocity field

To clarify the applicability ranges and limitations of 2D and 3D models for depth averaged velocity fields, the magnitude of each terms in depth averaged velocity equation are discussed by shallowness parameter in this section. Equations for horizontal depth averaged velocity are derived by depth-integrating momentum equations in horizontal directions without the assumption of hydrostatic pressure distribution.

$$\rho\left(\frac{\partial U_i h}{\partial t} + \frac{\partial U_i U_j h}{\partial x_j}\right) = -\rho g h \frac{\partial z_b}{\partial x_i} - \tau_{bi} - \left(\frac{\partial \rho g h^2/2}{\partial x_i} + \frac{\partial h d p_0}{\partial x_i} + d p_b \frac{\partial z_b}{\partial x_i}\right) + \frac{\partial h \tau_{ij}}{\partial x_j} \tag{3}$$

Here, τ_{bi}: bed shear stress, dp: pressure deviation from hysrostatic pressure distribution ($p=\rho g(z_s-z)+dp$), dp_0: depth averaged dp, dp_b: dp on bottom, τ_{ij} :horizontal shear stress due to turbulence $\nu_t S_{ij}$ and vertical velocity distribution $\overline{u_i'u_j'}$, S_{ij}: depth averaged strain velocity, u_i': velocity deviation from depth averaged velocity.

Expressing representative horizontal scale L_0 , water depth (vertical scale) h_0 and velocity U_0, physical quantities in equation (3) are non-dimensionalized as

$$\begin{aligned}&U_i=(U_i)^*U_0,\quad h=(h)^*h_0,\quad x_i=(x_i)^*L_0,\quad \tau_{0i}/\rho=(\tau_{0i})^*U_0{}^2\varepsilon_*{}^2,\quad dp=(dp)^*U_0{}^2\varepsilon_s{}^2,\\ &\nu_t S_{ij}=\varepsilon_0{}^{1.2}U_0{}^2\varepsilon_*\varepsilon_s(\nu_t)^*(S_{ij})^*,\quad \overline{u_i'u_j'}=\varepsilon_0{}^{-0.5}\varepsilon_*{}^2U_0{}^2(\overline{u_i'u_j'})^*\end{aligned} \tag{4}$$

Here, $\varepsilon_*=u_{*0}/U_0$, u_{*0}: representative shear velocity, $\varepsilon_0=0.1$. ν_t is evaluated by zero-equation model, $\nu_t=\kappa u_* h/6$. $\overline{u_i'u_j'}$ is evaluated by vertical velocity distribution of quadratic curve (Uchida & Fukuoka, 2009). By substituting equation (4) into equation (3), equation (5) is derived.

$$F_z+\left(\frac{\tau_0}{h}\right)^*+\frac{\varepsilon_s}{\varepsilon_*{}^2}\left(F_c+\frac{1}{Fr^2}F_h\right)+\frac{\varepsilon_s{}^3}{\varepsilon_*{}^2}F_{dp}+Fr^2\varepsilon_s{}^2\left(\frac{dp_b}{h}\right)^*+\frac{\varepsilon_0{}^{1.2}\varepsilon_s{}^2}{\varepsilon_*}F_t+\frac{\varepsilon_s}{\varepsilon_0{}^{0.5}}F_{c\Delta}=0 \tag{5}$$

Here, F_z=gravity terms, F_c: convection terms, F_h, F_{dp}: pressure terms of hydrostatic, non-hydrostatic component, F_t, $F_{c\Delta}$: horizontal shear stress term due to turbulence, vertical velocity distribution, $Fr^2=U_0{}^2/gh_0$.

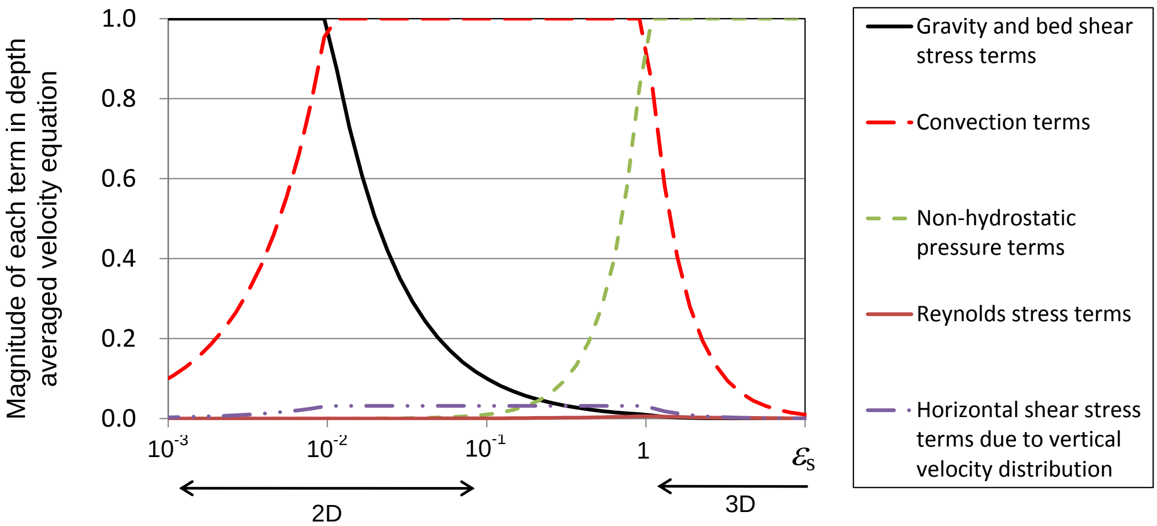

Figure 1: Magnitude of each term in depth averaged velocity equation

Figure 1 shows magnitude of each term in equation (5) by shallowness parameter ε_s for $\varepsilon_*=0.1$, setting the maximum magnitude for ε_s to 1 in the vertical axis. For very shallow water flow $\varepsilon_s<10^{-3}$, depth averaged velocity field is determined only by the gravity and the bed shear stress terms without influence of the convection terms. In this range, we can apply formulae for uniform flows such as the Chezy formula or the Mannig formula to evaluate depth averaged velocity. The convection term is dominant for $10^{-3}<\varepsilon_s<1$, in which a depth integrated model is useful. However, the non-hydrostatic terms become larger from $\varepsilon_s=0.1$. So, for $0.1<\varepsilon_s<1$, a non-hydrostatic depth integrated model (quasi-3D model) is required, while 2D model is useful for $10^{-3}<\varepsilon_s<0.1$. A 3D model is required for small scale phenomena of $1<\varepsilon_s$, in which the convection terms become smaller with increasing ε_s. The horizontal shear stress terms due to turbulence and vertical velocity distribution are small regardless of shallowness parameter, because the magnitudes of these terms are evaluated for uniform flow condition in this study.

3.2 Bottom velocity field

A calculation method of bottom velocity has an important role for a bed variation analysis. A non-dimensional equation of (1) is

$$(u_b)^* = (u_s)^* + \varepsilon_0^{-0.8}\varepsilon_*(\Omega h)^* + \varepsilon_s^{\ 2} F_w \tag{6}$$

Here, F_w: spatial difference terms in vertical velocity of equation (1).

The equation concerning a water surface velocity is derived from momentum equation in the very thin layer δz_s under the water surface (Uchida & Fukuoka, 2010, 2011).

$$\frac{\partial u_{si}}{\partial t} + u_{sj}\frac{\partial u_{si}}{\partial x_j} = -g'\frac{\partial z_s}{\partial x_i} + P_{si} \tag{7}$$

$$P_{si} = \frac{2\nu_t}{h^2}\{12(u_{sei} - u_{si}) - (3\delta u_i - 6\Delta u_i)\} \tag{8}$$

Here, $g' = g + (1/\rho)(\partial dp/\partial z)$, P_{si} :shear stress acting on undersurface of δz_s, $u_{sei} = U_i + (\delta u_i - \Delta u_i)$, δu_i: $u_{si} - u_{bi}$, Δu_i: $u_{si} - U_i$. According to equation (4), non-dimensional equation for water surface is given by

$$(u_s)^* = (u_{se})^* + \frac{\varepsilon_0^{\ 0.2}\varepsilon_s}{\varepsilon_*}\left\{F_{sc} + F_{s\tau} + \frac{F_h}{Fr^2}(1 + Fr^2\varepsilon_s^{\ 2}F_{sr})\right\} \tag{9}$$

Here, F_{sc}: convection terms, $F_{s\tau}$: shear stress terms due to non-equilibrium vertical velocity distribution, F_{sr}: curvature terms of water surface.

A depth averaged horizontal vorticity equation is derived from depth integrated horizontal vorticity equation (Uchida & Fukuoka, 2009, 2010, 2011).

$$\frac{\partial \Omega_i h}{\partial t} = ER_{\sigma i} + P_{\omega i} + \frac{\partial h D_{\omega ij}}{\partial x_j} \tag{10}$$

Here, $ER_{\sigma i} = u_{si}\omega_{s\sigma} - u_{bi}\omega_{b\sigma}$, $P_{\omega i} = C_{p\omega}\nu_{tb}(\omega_{bei} - \omega_{bi})/h$, $C_{p\omega} = \kappa/\alpha$, $\alpha = \kappa/6$, ν_{tb}: vortex viscosity on bottom converted to depth averaged value,

$$D_{\omega ij} = -U_j\Omega_i + U_i\Omega_j + \overline{\omega'_j u'_i} - \overline{\omega'_i u'_j} + \frac{\nu_t}{\sigma_\omega}\frac{\partial \Omega_i}{\partial x_j}$$

The depth averaged vorticity equation (10) is non-dimentionalized as

$$(\Omega h)^* = (\Omega_e h)^* + \varepsilon_*^{-1}\varepsilon_0^{-0.1}\varepsilon_s\left(F_{\omega 1} + \varepsilon_*\varepsilon_0^{-0.8}F_{\omega 2} + \varepsilon_0^{\ 1.2}\varepsilon_*\varepsilon_s F_{\omega 3}\right) \tag{11}$$

Here, $F_{\omega 1}$, $F_{\omega 2}$ and $F_{\omega 3}$ are vorticity transport terms due to depth-averaged, vertical velocity distribution and turbulent diffusion.

Substitution of equation (9) and (11) into equation (6) leads to equation (12) for bottom velocity pattern.

$$\begin{aligned}(u_b)^* = (U)^* &+ \varepsilon_0^{\ 0.2}\frac{\varepsilon_s}{\varepsilon_*}\left\{F_{sc} + F_{s\tau} + \frac{F_h}{Fr^2}\left(1 + Fr^2\varepsilon_s^{\ 2}F_{sr}\right)\right\} \\ &+ \frac{\varepsilon_s}{\varepsilon_0^{\ 0.9}}\left(F_{\omega 1} + \varepsilon_* F_{\omega 2} + \varepsilon_s\varepsilon_*\varepsilon_0^{\ 1.2}F_{\omega 3}\right) + \varepsilon_s^{\ 2}F_w\end{aligned} \tag{12}$$

Figure 2 shows magnitude of each term in the bottom velocity equation (12) for ε_*=0.1. For ε_s<10^{-2}, because bottom velocity field is determined only by depth averaged velocity field, a 2D model is applicable to evaluate bottom velocity for bed variation analysis. On the other hand, for $10^{-2} < \varepsilon_s$, the convection terms of water surface velocity and depth averaged horizontal vorticity become larger with increasing ε_s. So, one cannot apply a 2D model to bottom velocity analyses for $10^{-2} < \varepsilon_s$, ignoring variations in vertical velocity distribution. The upper limit $\varepsilon_s=10^{-2}$ of the applicable range of a 2D model in Figure 2 is smaller than that of Figure 1 ($\varepsilon_s=10^{-1}$). The results indicate that a quasi-3D model plays a more significant role in bed variation analysis. For 0.1 < ε_s, because the magnitude of spatial difference terms in vertical velocity becomes larger with increasing ε_s, the shallow water assumption is unsuitable. For 1 < ε_s , the depth averaged velocity has little influence in bottom velocity field any more.

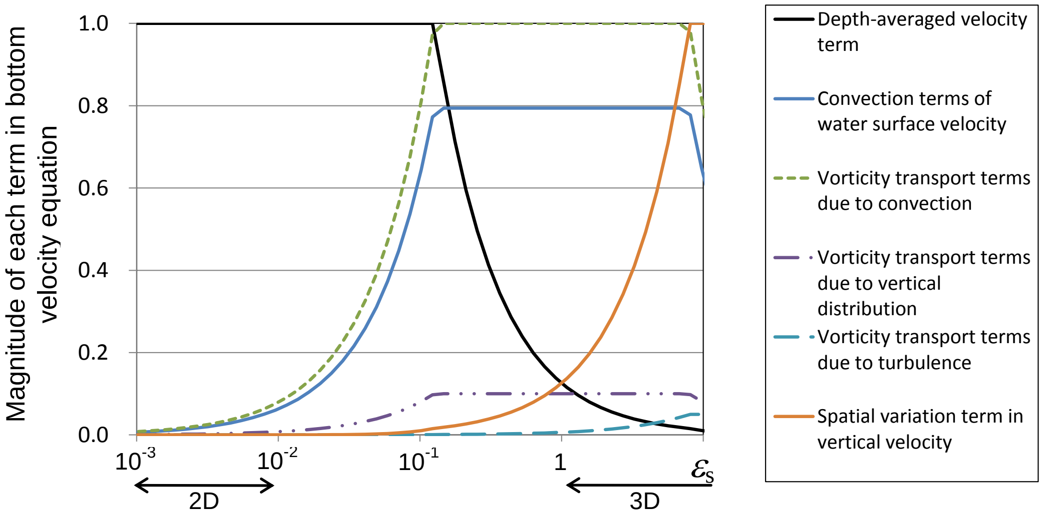

Figure 2: Magnitude of each term in bottom velocity equation

The above results of Figure 1 and 2 indicate that a depth integrated model without assumption of shallow water flow such as non-hydrostatic pressure distribution and spatial difference in vertical velocity is required for 0.1 < ε_s < 1.

4. General Bottom Velocity Computation Method without Shallow Water Assumption

4.1 Governing Equation for Depth Averaged Vertical Velocity Equation

To calculate bottom velocity by equation (1) without the assumption of a shallow water flow condition, a Poisson equation for time variation in depth integrated vertical velocity is derived as follows. Unknown quantities in the BVC method are computed by time advance method. Unknown quantities at n+1 step are required to satisfy

$$\delta u_i^{\,n+1} = \varepsilon_{ij3}(\Omega_j h)^{n+1} + \frac{\partial (Wh)^{n+1}}{\partial x_i} - w_s^{\,n+1}\frac{\partial z_s^{\,n+1}}{\partial x_i} + w_b^{\,n+1}\frac{\partial z_b^{\,n+1}}{\partial x_i} \tag{13}$$

Here, predicted velocity difference between water surface and bottom velocity $\delta u_i^{\,p}$ by previous depth integrated vertical velocity Wh^n at n step is introduced:

$$\delta u_i^{\,p} = \varepsilon_{ij3}(\Omega_j h)^{n+1} + \frac{\partial (Wh)^{n}}{\partial x_i} - w_s^{\,n+1}\frac{\partial z_s^{\,n+1}}{\partial x_i} + w_b^{\,n+1}\frac{\partial z_b^{\,n+1}}{\partial x_i} \tag{14}$$

By subtracting equation (14) from equation (13), we have

$$\varepsilon(\delta u_i) = \frac{\partial \phi}{\partial x_i}, \quad \varepsilon(\delta u_i) = \delta u_i^{\ n+1} - \delta u_i^{\ p}, \quad \phi = Wh^{n+1} - Wh^n \tag{15}$$

On the other hand, corrected vertical velocity $\varepsilon(Wh)=(Wh)^{n+1}-(Wh)^p$ is represented as follows. $(Wh)^p$ is defined as depth integrated vertical velocity by using predicted horizontal velocity (14). Vertical velocity at arbitraly depth η $(=(z_s-z)/h)$ is

$$w_\eta = w_{\sigma\eta} + u_{j\eta}\frac{\partial z_\eta}{\partial x_j}, \quad w_{\sigma\eta} = -\frac{\partial}{\partial x_j}\left(\int_{z_b}^{z_\eta} u_j dz\right) \tag{16}$$

And the vertical velocity distribution is corrected by $\varepsilon(\delta u_i)$:

$$\varepsilon(u_i) = \varepsilon(\delta u_i)\left(-4\eta^3 + 3\eta^2\right) \tag{17}$$

By substituting equation (17) into equation (16) and making depth-integration, corrected vertical velocity $\varepsilon(Wh)$ is obtained as

$$\varepsilon(Wh) = k_1 \frac{\partial h^2 \varepsilon(\delta u_j)}{\partial x_j}, \quad k_1 = 1/20 \tag{18}$$

Finally, the Poisson equation for vertical velocity variation in time is derived from substitution of equation (15) into equation (18).

$$k_1 \frac{\partial}{\partial x_j}\left(h^2 \frac{\partial \phi}{\partial x_j}\right) + \phi^P - \phi = 0, \quad \phi^P = (Wh)^P - (Wh)^n \tag{19}$$

4.2 Governing Equation for Non-Hydrostatic Pressure Distribution

Depth integrated vertical momentum equation is used to determine non-hydrostatic bottom pressure distributions.

$$\frac{\partial hW}{\partial t} + \frac{\partial hWU_j}{\partial x_j} = \frac{dp_b}{\rho} - \tau_{bj}\frac{\partial z_b}{\partial x_j} + \frac{\partial h\tau_{zj}}{\partial x_j} \tag{20}$$

For simplicity, the unsteady and horizontal shear stress terms are neglected and the linear vertical pressure distribution is assumed:

$$dp = \eta dp_b \tag{21}$$

5. Application to Flow Fields around a Cylinder and Discussion

The general BVC method without the shallow water assumption is applied to a flow around a cylinder mounted on fixed rough bed, which was measured by Roulund *et al.*(2005). Table 1 shows the experimental conditions and Figure 3 shows computational gird around a cylinder made of 1/40 of the diameter. Experimental discharge is given at upstream end and water depth is given at downstream end.

Table 1: experimental conditions by Roulund et al.(2005)

Diameter D (m)	0.536
Depth h_0 (m)	0.54
Cannel width B (m)	4.0
Fr number	0.14

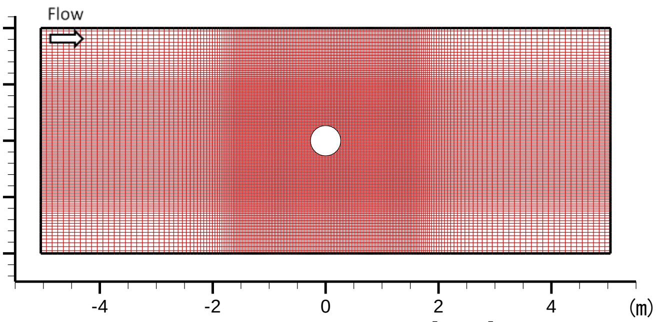

Figure 3: Computational grid

Figure 4 shows computed results of (a) depth-averaged vertical velocity, (b) non-hydrostatic bottom pressure and (c) bottom velocity distributions. Downward and reverse bottom flows are induced with low pressure in front of a cylinder and high pressure in both sides of a cylinder. Those are typical characteristics of a horse shoe vortex. Figure 5 shows comparison between experimental and calculational results by the general BVC method for velocity distributions on longitudinal cross-sections along the center line of a cylinder. It is demonstrated that the method can simulate 3D flow field in front of a structure despite of the depth integrated model.

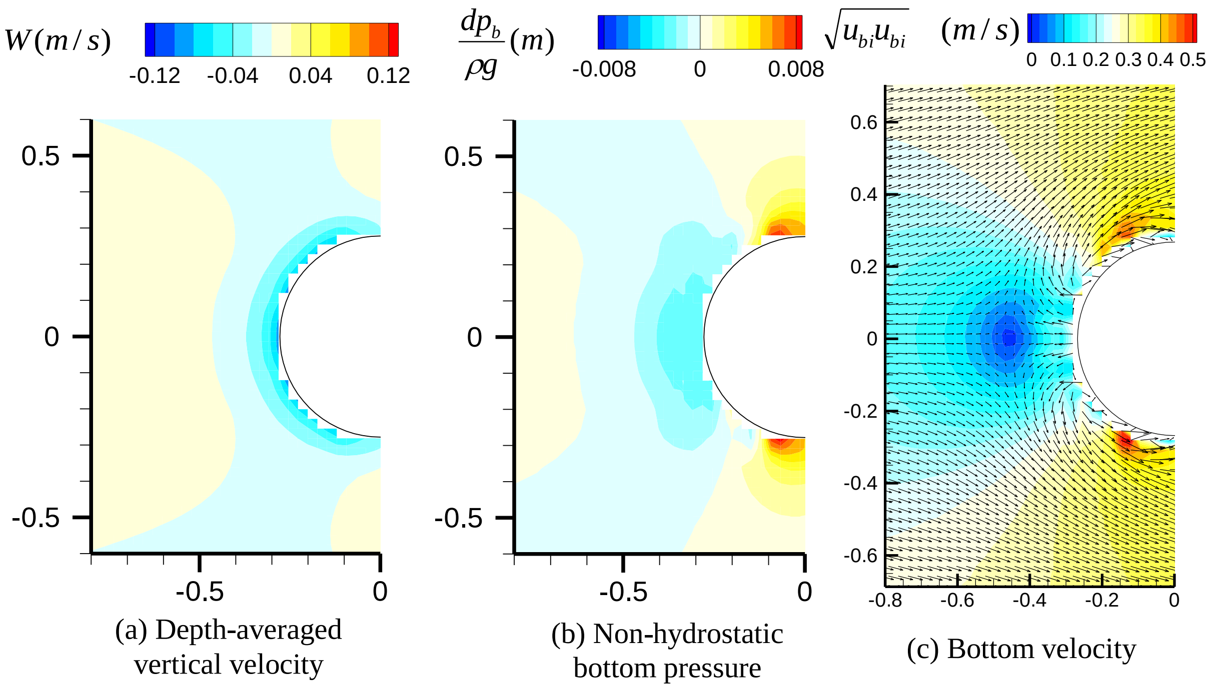

Figure 4: Calculated results by the general BVC method to experimental condition by Roulund et

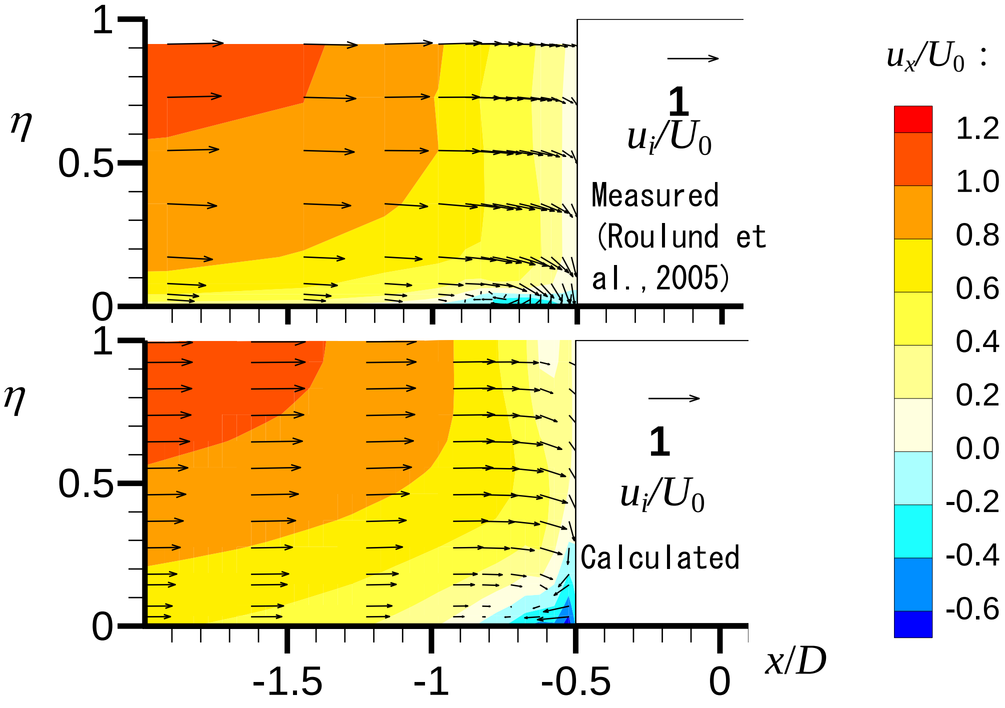

Figure 5: Velocity distribution on the central vertical-section in front of a cylinder.

6. Conclusions

Applicability of various computational methods for a target phenomenon is discussed based on momentum equations. It is clarified that a depth integrated model without the assumption of shallow water flow is required to calculate both of depth averaged and bottom velocity flow fields for $0.1 < \varepsilon_s < 1$.

To develop the general bottom velocity computation (BVC) method without shallow water assumptions, the Poisson equation for vertical velocity variation in time is derived. And computation method for non-hydrostatic pressure distribution by using the depth-integrating vertical momentum equation is presented.

The general BVC method is applied to flow field around a cylinder mounted on the fixed and rough bed. It is demonstrated that the general BVC method can calculate 3D flow structures around a cylinder through the comparison with the experimental results of Roulund et al. (2005).

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